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Towards a common thread in complexity: an accuracy-based approach

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Abstract

The complexity of a system, in general, makes it difficult to determine some or almost all matrix elements of its operators. The lack of accuracy acts as a source of randomness for the matrix elements which are also subjected to an external potential due to existing system conditions. The fluctuation of accuracy due to varying system conditions leads to a diffusion of the matrix elements. We show that, for single-well potentials, the diffusion can be described by a common mathematical formulation where system information enters through a single parameter. This further leads to a characterization of physical properties by an infinite range of single-parametric universality classes.

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1. Introduction

For systems that can be described mathematically, physical information can be derived, in principle, from detailed knowledge of the operators that govern their evolution. Physical systems can however be complex in nature and it is not always possible to determine the operator exactly or, even if they are known, to solve the equations they determine. This paper aims to model the statistical behaviour of those complex systems where a matrix representation of the operators is meaningful.

The complexity may appear in various forms, for example, as noise due to many-body interactions or an external disorder potential, as chaos due to scattering of a particle from boundaries (e.g. clean quantum dots) and as coherence patterns emerging out of randomness (see, for example, [1] for various definitions of complexity). For example, consider the Hamiltonian of a many-body system. If the local interactions are complicated in a specific part of the system, the evaluation of the corresponding matrix elements becomes technically difficult. These elements can then be determined only within a certain degree of accuracy and can best be described by a probability density. However, the system may also contain parts where interactions are simple and the related matrix elements can exactly be calculated. The operator then turns out to be a matrix with both random and non-random elements; we refer

such a matrix as a generalized random matrix. Similar matrices would also appear for systems containing a combination of chaotic as well as ordered components. The properties of such systems can then be modelled by an ensemble of generalized random matrices.

In recent years, due to an increasing degree of complexity in systems of industrial and technological interests, the mathematical models such as random matrix ensembles have become necessary. In fact, a particular class of these ensembles, known as stationary ensembles [2], has been successfully applied for modelling of the operators for a wide range of complex systems, e.g. nuclei, atoms, molecules, disordered and chaotic systems, quantum chromodynamics, elastomechanics, electrodynamics (see reviews [2–11] and references therein for details), mathematical areas such as Riemann zeta function, enumeration problems in geometry and fluctuations in random permutations [12], biological systems [13], stock markets [14], atmospheric sciences [15], complex networks [16], etc[1]. The stationary random matrix ensembles are basis-invariant ensembles, characterized by a similar and independent distribution of almost all elements [2]. This restricts their applicability only to the generators with wavefunctions extended in the entire system or with a coherent scattering of waves. However, the matrix elements distribution can significantly be affected by various system conditions, e.g. missing interactions among some of the sub-systems, a variation in their degree or nature, symmetry and boundary conditions, dimensionality, disorder, etc [21]. These conditions may result in different strengths of the elements, correlations between them and localized waves; the corresponding ensembles are then basis-non-invariant.

The presence of local interactions and wave-localization phenomena is quite generic to a wide range of complex systems. The statistical analysis of their physical properties requires therefore a search for new mathematical tools. The present study is an attempt in this direction. The basic idea here is to take into account the inaccuracy in the matrix representation of an operator of a complex system. The fluctuation of accuracy with changing system conditions results in a change of distribution parameters of various matrix elements. This leads to a seemingly multi-parametric diffusion of the ensemble density. However, as shown here, the diffusion is essentially governed by a single parameter only. The information can then be utilized to study the fluctuations of the physical properties due to varying system conditions and express the results in a common mathematical form for a wide range of complex systems.

The statistical behaviour of complex systems and the possibility of a common mathematical formulation were recently studied by considering their maximum entropy models [18, 19]. The latter is based on the formulation of the ensemble density by maximizing the information entropy under constraints imposed on the system [17]. The ensemble density is then utilized to extract the distribution of eigenvalues and eigenfunctions and desired physical information. The maximum entropy approach indicated the possibility of a classification of the complex systems into various universality classes (based on the behaviour of their statistical measures and characterized by the complexity parameter [18, 19]). These results were also verified numerically for certain cases [20, 21, 25]. However, the complexity parameter formulation within this approach gives rise to some queries which required a more intuitive physical reasoning for their resolution. This motivates us to consider the accuracy-based approach which not only resolves the queries but also helps in the generalization of the single-parametric formulation to a wider range of complex systems (those with system conditions subjecting matrix elements to a potential with a single minima only).

The paper is organized as follows. Section 2 describes the diffusive dynamics for the matrix elements of a Hermitian operator subjected to an external potential of type $e^{-V(H)}$

[1] Many new applications and several other references of random matrix theory can be found by searching the cond-mat archives.

as well as random noise originated in the complexity of the system. (We have considered here the real-symmetric case only, however the results are valid for the complex Hermitian and real-quaternion cases too.) The comparison of this approach with the maximum entropy approach is discussed in section 3. This is followed by section 4 describing the derivation of the statistical measures of the eigenvalues and eigenfunctions using standard perturbation theory. Note, for generic potentials, the derivation of the measures by a direct integration of the evolution equation (the method used for Gaussian cases in [20]) is technically difficult. We conclude in section 5 with a summary of our main results.

2. Accuracy-driven diffusion of matrix elements

Consider, as an example, a Hermitian operator H of a complex system with time-reversal symmetry and integer angular momentum. It is possible to choose a generic basis, say $|\phi_k\rangle$ ($k = 1 \to N$), preserving the time-reversal symmetry for the matrix representation of H; the matrix turns out to be real symmetric in this basis with its elements $H_{kl} = \langle\phi_k|H\phi_l\rangle$. For notational simplification, let us denote them by H_μ, where $\mu \equiv \{kl; s\}$ is a single index which can take a value from 1 to M ($M = N(\beta N - \beta + 2)/2$, the number of independent matrix elements). Here, β is the number of components of H_μ; thus, $\beta = 1$ for the real-symmetric case.

Due to the presence of complicated interactions in the system, it is technically difficult to evaluate some/all elements of the operator matrix in a generic basis. Consequently, the matrix elements can be determined only within a certain degree of accuracy which, being sensitive to local system conditions, varies from element to element. The accuracy fluctuates rapidly as the system conditions change, with different 'time-scale' of fluctuations for each matrix element. The variation of an element H_μ with changing system conditions can therefore be mimicked by a particle undergoing Brownian dynamics due to rapidly fluctuating forces in addition to an external force (due to existing system conditions). The matrix elements of a physical system also have a natural tendency to oppose the cause for their change. The dynamics is therefore subjected to a local frictional force too.

Consider the 'particle' H_μ in equilibrium under the external force $V(H_\mu)$ due to existing system conditions. The equation of motion for H_μ due to changing system conditions can be written as

$$\frac{\mathrm{d}^2 H_\mu}{\mathrm{d}t_\mu^2} = -f\frac{\mathrm{d}H_\mu}{\mathrm{d}t_\mu} + V(H_\mu) + A(t_\mu), \tag{1}$$

where f is the friction coefficient and $A(t_\mu)$ is a rapidly fluctuating force in 'time' t_μ (a pseudo time only, a measure of the scale for accuracy fluctuations) with the following usual properties:

$$\langle A(t_{\mu 1})A(t_{\mu 2})\cdots A(t_{\mu(2n+1)})\rangle = 0, \tag{2}$$

$$\langle A(t_{\mu 1})A(t_{\mu 2})\cdots A(t_{\mu(2n)})\rangle = \sum_{\text{pairs}}\langle A(t_{\mu i})A(t_{\mu j})\rangle\langle A(t_{\mu k})A(t_{\mu l})\rangle\cdots, \tag{3}$$

$$\langle A(t_{\mu i})A(t_{\mu j})\rangle = (2/f)\delta(t_{\mu i} - t_{\mu j}), \tag{4}$$

where $\langle\cdot\rangle$ refers to the ensemble average, $t_{\mu j}$ refers to the jth step in the time-scale t_μ and the summation in equation (3) extends over all distinct ways in which the $2n$ indices can be divided into n pairs. Further, for a clear exposition of the ideas, we consider here the potential $V(H_\mu)$ as a function of H_μ with a single minima.

The Langevin equation can now be integrated: let H_μ be the position of the particle at time t_μ which changes to position $H_\mu + \delta H_\mu$ at a later time $t_\mu + \delta t_\mu$ (here, t_μ is chosen to be

long enough for the effects of initial velocity to become negligible). Due to the presence of rapidly fluctuating forces, the variation δH_μ in the position of the particle will behave like a random variable. Using equations (1)–(4) and keeping terms only of first order in δt_μ, one gets

$$f\langle \delta H_\mu \rangle = -V(H_\mu)\delta t_\mu, \qquad f\langle (\delta H_\mu)^2 \rangle = (g_\mu/\beta)\delta t_\mu, \tag{5}$$

with $g_\mu \equiv g_{kl} = 1 + \delta_{kl}$. Due to random variations in particle position with changing system conditions, it is appropriate to consider a time-dependent probability density $\rho_\mu(H_\mu, t_\mu)$ that the particle will be at the position H_μ at time t_μ. Assuming a Markovian process (that is the independence of future evolution from past states, dependence only on the present state), one can write

$$\rho_\mu(H_\mu; t_\mu + \delta t_\mu) = \int \rho_\mu(H_\mu - \delta H_\mu; t_\mu)\rho_{\text{cond}}(H_\mu - \delta H_\mu; \delta H_\mu; \delta t_\mu)\, \mathrm{d}\delta H_\mu, \tag{6}$$

where ρ_{cond} is the conditional probability that the position of the particle changes from $H_\mu - \delta H_\mu$ to H_μ in a time interval δt_μ. Expanding both sides of equation (6) in a power series of δH_μ and δt_μ and subsequently using equation (5), we get (in the limit $\delta t \to 0$)

$$f\frac{\partial \rho_\mu}{\partial t_\mu} = \frac{\partial}{\partial H_\mu}\left[\frac{g_\mu}{2\beta}\frac{\partial}{\partial H_\mu} + V(H_\mu)\right]\rho_\mu. \tag{7}$$

Equation (7) describes the evolution of H_μ with respect to the time-scale t_μ which in turn depends on the time-scale for accuracy fluctuations (and therefore system conditions) surrounding H_μ. For systems where the coupling of any two basis states through the generator H is independent of the coupling between other states (i.e. all matrix elements are independent of each other), the fluctuations in the accuracy of each matrix element are independent too. Each element can therefore be assumed to be subjected to a random force fluctuating at a time-scale independent of others (that is, all t_μ independent of each other). This gives us M equations, of type (7), for the independent evolutions of M elements H_μ.

The joint probability distribution $\rho(\{H_\mu\}; \{t_\mu\})$ of all matrix elements can now be defined as

$$\rho(\{H_\mu\}; \{t_\mu\}) = \prod_\mu \rho_\mu(H_\mu; t_\mu) \tag{8}$$

which along with equation (7) leads to the equation for multi-parametric evolution of ρ:

$$f\sum_\mu \frac{\partial \rho}{\partial t_\mu} = \sum_\mu \frac{\partial}{\partial H_\mu}\left[\frac{g_\mu}{2\beta}\frac{\partial}{\partial H_\mu} + V(H_\mu)\right]\rho. \tag{9}$$

For a system undergoing evolution as a whole unit, it is natural to seek a common scale, say τ, at which all its constituents, i.e. matrix elements, vary simultaneously. Let us therefore consider the evolution of ρ with respect to τ. Assuming again a Markovian process, we have

$$\rho(\{H_\mu\}; \tau + \delta\tau) = \int \rho(\{H_\mu\} - \delta\{H_\mu\}; \tau)\rho_{\text{cond}}(\{H_\mu\} - \delta\{H_\mu\}; \delta\{H_\mu\}; \delta\tau)\, \mathrm{D}\delta H, \tag{10}$$

where $\mathrm{D}\delta H \equiv \prod_\mu \mathrm{d}\delta H_\mu$. Expanding both sides of equation (10) in a power series of δH_μ and $\delta\tau$, we get (in the limit $\delta t \to 0$)

$$\frac{\partial \rho}{\partial \tau}\delta\tau = \sum_\mu \frac{\partial}{\partial H_\mu}\left[\frac{\partial}{\partial H_\mu}\frac{\langle (\delta H_\mu)^2\rangle}{2} - \langle \delta H_\mu \rangle\right]\rho. \tag{11}$$

As both equations (11) and (9) describe the evolution of the probability density of H, they should be analogous. A comparison of the equations then gives the conditions

$$\frac{\partial \rho}{\partial \tau}\delta\tau = \sum_\mu \frac{\partial \rho}{\partial t_\mu}\delta t_\mu \tag{12}$$

and

$$f\langle \delta H_\mu \rangle = -V(H_\mu)\delta\tau, \qquad f\langle (\delta H_\mu)^2 \rangle = (g_\mu/\beta)\delta\tau. \tag{13}$$

The two conditions imply

$$\delta\tau = \delta t_1 = \delta t_2 = \cdots = \delta t_{\tilde{M}}. \tag{14}$$

This is satisfied if τ is defined as $\tau = \frac{\sum_{\mu=1}^{N} a_\mu t_\mu}{\sum_\mu a_\mu}$, with a_μ as arbitrary constants. However, physical reasoning (based on no preference by random forces to any particular component of the system) suggests us to choose a_j equal.

The solution of equation (11) for an arbitrary initial condition, say H_0 at $\tau = \tau_0$ can be given as

$$\rho(H, \tau | H_0, \tau_0) \propto \exp[-(\alpha/2\beta)\,\mathrm{Tr}(H - \eta H_0)^2], \tag{15}$$

with $\alpha = (1-\eta^2)^{-1}$ and $\eta = \mathrm{e}^{-(\tau-\tau_0)/f}$. The probability density of H can now be extracted by integrating over an ensemble of initial conditions. Although equations (11) and (15) are derived for the case $\beta = 1$, it is easy to show, following essentially the same steps, their validity for the complex Hermitian case $\beta = 2$ and the real-quaternion case $\beta = 4$.

Note the accuracy scales τ_μ depend on local system conditions which can vary from system to system. However, as equations (11) and (15) indicate, $\rho(H)$ is insensitive to the details of the local system conditions; it depends only on their average behaviour described by τ besides global constraints, e.g. $V(H)$ and symmetry conditions (note $V(H)$ has no explicit dependence on τ_μ). Thus, analogous to their maximum entropy models, the accuracy-based approach indicates a single-parametric dependence of the density $\rho(H)$ for simple harmonic confinement $V(H) = H$. It further generalizes the formulation to the systems with conditions giving rise to a generic single-well (single-minima) potential. The approach can, in principle, be extended to the multi-well potentials too, however it requires a modification of the technical details. We intend to pursue these cases in the near future.

It is important to note that the form of equation (11) for the case $V(H) = H$ is analogous to Dyson's Brownian model [2, 22]. The latter deals with the case of a stationary ensemble subjected to a random perturbation. However, the Brownian dynamics of matrix elements in the accuracy model is different from Dyson's case; there are two main differences:

(1) In Dyson's model, the randomness caused due to a perturbation is the same for almost all the matrix elements. In the accuracy model, the origin of randomness is the lack of accuracy which is sensitive to local conditions. Different matrix elements may therefore be subjected to different randomness.
(2) In Dyson's model, the evolution occurs due to a variation in the perturbation strength and is single parametric. In the accuracy model, the evolution is brought by the fluctuating accuracy due to varying system conditions. As a consequence, we need to consider a multi-parametric evolution of probability density (unlike the single-parametric evolution in Dyson's case). However, as equation (11) indicates, the multi-parametric evolution can be reduced to a single-parametric evolution.

3. A comparison of the accuracy-based approach and the maximum entropy approach

The objective of this section is to indicate the analogy of the results obtained by the accuracy model and maximum entropy models of complex systems notwithstanding their seemingly different origins. For a clear comparison, we briefly review the maximum entropy approach. This approach is based on the representation of a complex system by an ensemble of matrices; here the probability density of the matrix elements is formulated by maximizing the information

entropy under known system constraints (see [17] for details). However, the density in the accuracy-based approach is obtained as a non-stationary state of a diffusion process. This can further be clarified by an example. The accuracy model leads to a Gaussian density $\rho(H)$ if $V(H) = H$ and the initial density is Gaussian too (see equation (15)). However, the maximum entropy theory leads to a Gaussian density if the available information about matrix elements is limited to their average behaviour and variances only:

$$\rho(H, v, b) = \prod_{\mu} \rho_{\mu}(H_{\mu}, v_{\mu}, b_{\mu}) = C \exp\left[-\sum_{\mu}(1/2v_{\mu})(H_{\mu} - b_{\mu})^2\right], \quad (16)$$

with C as a normalization constant, v, b as the matrices of variances v_{μ} and mean b_{μ}, respectively, and the symbol $\sum_{\mu}$ implying a summation over the independent matrix elements only.

The emergence of the single-parametric formulation in the maximum entropy approach can briefly be explained as follows. The Gaussian nature of ρ_{μ} (see equation (16)) leads to a relation among its derivatives with respect to $H_{\mu}, v_{\mu}, b_{\mu}$:

$$T_{\mu}\rho = L_{\mu}\rho, \quad (17)$$

where

$$T_{\mu} = \left[(2/\tilde{g}_{\mu})x_{\mu}\frac{\partial\rho}{\partial v_{\mu}} - b_{\mu}\frac{\partial\rho}{\partial b_{\mu}}\right], \quad (18)$$

$$L_{\mu} = \frac{\partial}{\partial H_{\mu}}\left[\frac{g_{\mu}}{2\beta}\frac{\partial}{\partial H_{\mu}} + H_{\mu}\right]\rho, \quad (19)$$

with $x_{\mu} \equiv 1 - \tilde{g}_{\mu}v_{\mu}$, where $\tilde{g}_{\mu} \equiv \tilde{g}_{kl} = 2 - \delta_{kl}$ and g_{μ} is the same as in equation (5). A particular combination of the parametric derivatives $T = \sum_{\mu} T_{\mu}$ leads to a diffusion equation $T\rho = L\rho$ (with $L = \sum_{\mu} L_{\mu}$). The single-parametric formulation of the diffusion then follows by showing $T = \frac{\partial}{\partial Y}$, with Y as the complexity parameter [18, 19].

Equation (17) describes the evolution of ρ when all other matrix elements except H_{μ} are held fixed. It is therefore equivalent to equation (7) with $V(H_{\mu}) = H_{\mu}$ and ρ_{μ} replaced by ρ (following equation (8)). This implies

$$f\frac{\partial\rho}{\partial t_{\mu}} = (2/\tilde{g}_{\mu})x_{\mu}\frac{\partial\rho}{\partial v_{\mu}} - b_{\mu}\frac{\partial\rho}{\partial b_{\mu}}. \quad (20)$$

The scale t_{μ} can then be expressed in terms of the distribution parameters:

$$v_{\mu} = (1 - \mathrm{e}^{2t_{\mu}/f})/\tilde{g}_{\mu} + c_{1\mu}, \qquad b_{\mu} = \mathrm{e}^{t_{\mu}/f} + c_{2\mu}, \quad (21)$$

with $c_{1\mu}$ and $c_{2\mu}$ as constants specific to each v_{μ} and b_{μ}, respectively. This indicates the equivalence of $\tau = \sum_{\mu} t_{\mu}$ (the average scale for accuracy fluctuation) to the complexity parameter in the maximum entropy model (an average distribution parameter of the ensemble) [18, 19]. This further implies that the confinement by a simple harmonic force in the accuracy model is equivalent to the maximum entropy modelling of a system with known averages and variances of the matrix elements. Similarly, a general confining potential $V(H)$ in the accuracy model can be shown to be equivalent to a maximum entropy ensemble derived under the constraints $\langle U(H)\rangle =$ constant, where $U(H) = \int V(H)\,\mathrm{d}H$.

The equivalence of the accuracy approach to the maximum entropy approach can be used to clarify some of the points related to the latter. For example, in the maximum entropy approach, a particular combination $T\rho$ of the parametric derivatives leads to the Brownian-type diffusion; the reason to consider such a combination is not so obvious. However, the

accuracy approach clearly explains the reason: the combination is required to study the evolution of the system as a whole unit. Further, in the maximum entropy approach, the multi-parametric diffusion governed by the parameters Y_j, $j = 1, 2, \ldots, N$, is reduced to a single-parametric formulation by showing that all Y's except Y_1 are constants of evolution. However, in the accuracy-based approach, the single parameter existence follows from the necessity of the simultaneity of the dynamics of various matrix elements. As both approaches represent the same dynamics, this reconfirms the lack of any role played by the parameters $Y_2, \ldots, Y_M$ in the diffusion of matrix elements.

4. Diffusion of eigenvalues and eigenfunctions

The eigenvalue equation of a $N \times N$ Hermitian matrix H is given by $HU = U^{\dagger}E$ with E as the $N \times N$ diagonal matrix of eigenvalues, $E_{mn} = e_n\delta_{mn}$, and U as the $N \times N$ eigenvector matrix, unitary in nature: $U^TU = 1$ [2]. As described in [20], the statistics of the eigenvalues and/or eigenfunctions of H can be obtained from equation (17) by integrating over the eigenfunctions and eigenvalues, respectively; the results in [20], however, are valid only for $V(H)$ as a simple harmonic force. Here we apply second-order standard perturbation theory [26] to derive results for a more general form of $V(H)$; here again H is taken to be a real-symmetric matrix for simplification.

4.1. Eigenvalue statistics

A small change $\delta\tau$ in the parameter τ changes $\rho(H)$ and its eigenvalue statistics. By considering the matrix $H + \delta H$ in the diagonal representation of the matrix H, the change δe_n in the eigenvalues can be given as

$$\delta e_n = \delta H_{nn} + \sum_{m\neq n} \frac{|\delta H_{mn}|^2}{e_n - e_m} + o((\delta H_{mn})^3), \tag{22}$$

where $H_{mn} = e_n\delta_{mn}$ at the value τ of the parameter. This further gives (taking $f = I$ for simplification)

$$\langle \delta e_n \rangle = \langle \delta H_{nn} \rangle + \sum_{m=1, m\neq n}^{N} \frac{\langle |\delta H_{mn}|^2 \rangle}{e_n - e_m} \tag{23}$$

$$= \left[-V(e_n) + \sum_{m=1, m\neq n}^{N} \frac{1}{e_n - e_m} \right] \delta\tau. \tag{24}$$

Here, equation (24) has been obtained from equation (22) by using equation (13). Similarly, up to the first order of $\delta\tau$,

$$\langle \delta e_n \delta e_m \rangle = \langle \delta H_{nn} \delta H_{mm} \rangle = (2/\beta)\delta_{nm}\delta\tau. \tag{25}$$

The information about moments of the eigenvalues e_n can now be used to obtain their evolution equation. The theory of Brownian motion [27] informs us that the joint probability distribution $P(\{e_n\})$ for the eigenvalues e_n evolves with the increasing τ according to the Fokker–Planck equation,

$$\beta f \frac{\partial P}{\partial \tau} = L_E P, \tag{26}$$

$$L_E = \sum_{n=1}^{N} \frac{\partial}{\partial e_n} \left[\frac{\partial}{\partial e_n} + \sum_{m=1, m\neq n}^{N} \frac{\beta}{e_m - e_n} + \beta V(e_n) \right] P. \tag{27}$$

This equation describes the evolution of the eigenvalues of a complex system modelled by the ensemble $\rho(H)$ due to changing system conditions.

As in the case of the maximum entropy approach [18, 21, 23], the eigenvalue correlations for the case $V(e) = e$ can be obtained by using the analogy with Dyson's Brownian ensembles [2, 22]. For a general $V(e)$, the correlations can be analysed by mapping equation (26) to the Calogero–Sutherland Hamiltonian [23]. This can be achieved by using the transformation $\Psi = P/|Q_N|^{1/2}$ in equation (26) reducing it in a form $\frac{\partial \Psi}{\partial \tau} = -\hat{H}\Psi$, with $Q_N = \prod_{m\neq n}(e_m - e_n)^{\beta} \exp[-\beta \sum_n U(e_n)]$ and $U(e) = \int \mathrm{d}e V(e)$. The 'Hamiltonian' $\hat{H}$ turns out to be the Calogero–Sutherland Hamiltonian in one dimension [23]:

$$\hat{H} = -\sum_i \frac{\partial^2}{\partial e_i^2} + \sum_{i,j;i<j} \frac{\beta(2-\beta)}{(e_i - e_j)^2} + \beta \sum_i V(e_i). \tag{28}$$

Similar to the case $V(e) = e$ (see [18, 23] for details), the 'state' ψ or $P(\{e\}, \tau|H_0)$ for a generic $V(e)$ can be expressed as a sum over the eigenvalues and eigenfunctions of $\hat{H}$. The integration of the sum over the initial ensemble H_0 would then lead to the joint probability distribution $P(\{e\}, \tau)$ and thereby density correlations R_n for an unfolded spectrum (eigenvalues rescaled in the units of local spectral density). Note that the choice of the initial eigenvalue distribution at τ_0 depends on the global system constraints.

As equations (27) and (28) indicate, the confining potential $V(e)$ does not affect the short-range level correlations. The latter are governed only by the complexity parameter τ and underlying exact symmetry conditions. However, the long-range level correlations are sensitive to two factors: (i) the complexity parameter (i.e. the average accuracy fluctuation scale τ, or equivalently, the average distribution parameter) and (ii) the global system constraints, i.e. details of the external force $F(H)$ and the symmetry conditions. Thus, the systems subjected to *effectively* similar physical constraints will show analogous long-range correlations (after spectral unfolding) if their complexity parameters are equal. Here the term '*effectively* similar physical constraints' implies the similar symmetry conditions as well as the same mathematical form of the external potential although it may originate from different physical conditions. For example, a harmonic confinement of the matrix elements which also corresponds to their Gaussian distribution can be a physical characteristic of many systems related to different areas of physics.

4.2. Eigenfunction statistics

The evolution equation for the probability density of various eigenfunction components can similarly be obtained. Here again we consider the case of a real-symmetric operator for simplification. The eigenvector matrix $U \equiv O$ is then orthogonal: $O^T O = 1$ [2]. Using standard perturbation theory for Hermitian operators, the second-order change in the jth component O_{jn} of an eigenfunction O_n due to a small change $\delta\tau$ can be described as

$$\delta O_{jn} = \sum_{m\neq n} \frac{|\delta H_{mn}|}{e_n - e_m} O_{jm} + \sum_{m,m'\neq n}^{N} \frac{\bar{|}\delta H_{mn}||\delta H_{m'n}|}{(e_n - e_m)(e_n - e_{m'})} O_{jm}$$
$$- \sum_{m\neq n}^{N} \frac{\bar{|}\delta H_{mn}||\delta H_{nn}|}{(e_n - e_m)^2} O_{jm} - \frac{1}{2} O_{jn} \sum_{m\neq n}^{N} \frac{\bar{|}\delta H_{mn}|^2}{(e_n - e_m)^2}. \tag{29}$$

As equation (13) indicates, the matrix elements of H are uncorrelated. Furthermore, at τ, $H_{mn} = e_n\delta_{mn}$ (due to $H + \delta H$ being considered in the diagonal representation of H) which gives, following from equation (15), $\delta H_{mn} = -V(H_{mn})\delta\tau = -V(e_n\delta_{mn})\delta\tau$. Thus, $\delta H_{mn} = V(0)\delta\tau = 0$ for $m \neq n$ and $V(0) = 0$. The ensemble-averaged O_{jn} then has a nonzero contribution only from the last term of equation (29) (see equation (5)):

$$\langle \delta O_{jn} \rangle = -\frac{1}{2} \sum_{m=1, m\neq n}^{N} \frac{O_{jn}}{(e_n - e_m)^2} \delta\tau. \tag{30}$$

Note for cases where $V(H_{mn})$ is nonzero for $m \neq n$, the first term contributes too. Further, for cases where $V(0) \neq 0$ or matrix elements are correlated, the other terms may also contribute.

The second moment of the eigenvector components has a contribution only from the first term in equation (29) (up to first order in $\delta\tau$)

$$\langle \delta O_{jn} \delta O_{kn} \rangle = \sum_{m,m'\neq n}^{N} \frac{\langle |\delta H_{mn}||\delta H_{m'n}| \rangle}{(e_n - e_m)(E_n - E_{m'})} O_{jm} O_{km'} = (2\beta) \sum_{m=1, m\neq n}^{N} \frac{O_{jm} O_{km}}{(e_n - e_m)^2} \delta\tau. \tag{31}$$

As the moments for eigenfunction components depend on eigenvalues too, we can first write the diffusion equation for the joint probability density $P_{ef,ev}(e_1, e_2, \ldots, e_n; Y)$ of all the components of an eigenfunction and all eigenvalues:

$$\frac{\partial P_{ef,ev}}{\partial \tau} = (L_O + L_E) P_{ef,ev}, \tag{32}$$

where L_O and L_E refer to two parts of the Fokker–Planck operator corresponding to the eigenvalues and eigenfunction components. Here, L_E is given by equation (27) and

$$L_O = \sum_j \frac{\partial}{\partial O_{jn}} \left[1/2 \frac{\partial}{\partial O_{jn}} \langle (\delta O_{jn})^2 \rangle - \langle \delta O_{jn} \rangle \right]. \tag{33}$$

A substitution of the moments (equations (30) and (31)) in equation (33) followed by an integration of equation (32) over all eigenvalues except e_n will then lead to the evolution equation for the joint probability density $P_n(O_n, e_n; Y)$; the equation turns out to be the same as equation (18) given in [20] and can further be used to derive various correlation measures for an eigenfunction [20].

Equation (29) can also be used to derive the joint probability distribution of the components of different eigenfunctions; again, for the cases with $V(0) = 0$, the results, e.g. the single-parametric formulation in infinite size limit, turn out to be the same as given in [20].

5. Conclusion

In this paper, we have studied the dynamics of the matrix elements of an Hermitian operator of a complex system subjected to a single-well potential. The dynamics is diffusive due to random forces originating from accuracy fluctuations due to varying system conditions. The information is then applied to explore the statistical behaviour of the eigenvalues and eigenfunctions. Our analysis suggests a possible classification of complex systems in an infinite range of universality classes characterized just by the complexity parameter and the nature of global physical constraints. The constraints, e.g. unitary/anti-unitary symmetries, and confining potential on matrix elements seem to divide complex systems in various universality classes. Each such class can further be divided into many sub-classes characterized by their complexity parameters. Note the 'constraint' universality class of a system refers to the broad nature of its complexity (the finer details seem to be irrelevant).

However, its sub-universality class depends on the degree of complexity only (measured by the complexity parameter). This can be explained by the following examples. The standard Gaussian orthogonal ensemble (GOE), the power-law ensemble of real matrices, and the time-reversal Anderson ensemble belong to the same 'constraints' universality class in the above classification [20], although their complexity parameters, in general, are not equal (approaching infinity for GOE and finite in the other two cases). However, for the system parameters leading to the same finite value of the complexity parameter, the Anderson ensemble and power-law ensemble show the same statistics [20].

The accuracy approach described here is applicable, in its present form, only to the cases with independent matrix elements subjected to a single-well potential. The frequent occurrence of correlated elements or multi-well potentials among complex systems makes their analysis desirable too. A generalization to these cases requires a more involved technical analysis. However, our intuition suggests the possibility of a similar classification for these cases too. For example, for the multi-well potentials, the accuracy scales and their fluctuations are sensitive to local system details and can therefore vary from one branch to another. This would lead to a variation of diffusion scales (the average accuracy scale or complexity parameter) in different branches. Thus the statistical properties within a single branch would belong to a universality class characterized by the local complexity parameter. However, the universality classes in different branches need not be analogous. The above suggestion seems to be in accord with the already known results for invariant ensembles with multi-well potentials [28]. This encourages us to pursue a detailed analysis and extension to non-invariant ensembles of such cases in the near future.

For the correlated cases, the accuracy scales for various elements are no longer independent. However, a recent study of the maximum entropy models of a few correlated cases indicates the existence of the universality classes among them too [19]. It is desirable to explore the possibility of its generalization to a wider range of such cases.

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